Bregman storage functions for microgrid control

C. De Persis and N. Monshizadeh

Abstract—In this paper we contribute a theoretical framework that sheds a new light on the problem of microgrid analysis and control. The starting point is an energy function comprising the “kinetic” energy associated with the elements that emulate the rotating machinery and terms taking into account the reactive power stored in the lines and dissipated on shunt elements. We then shape this energy function with the addition of an adjustable voltage-dependent term, and construct so-called Bregman storage functions satisfying suitable dissipation inequalities. Our choice of the voltage-dependent term depends on the voltage dynamics under investigation. Several microgrid controllers that have similarities or coincide with dynamics already considered in the literature are captured in our incremental energy analysis framework. The twist with respect to existing results is that our incremental storage functions allow for a large signal analysis of the coupled microgrid. This obviates the need for simplifying linearization techniques, and for the restrictive decoupling assumption in which the frequency dynamics is fully separated from the voltage one. A complete Lyapunov stability analysis of the various systems is carried out along with a discussion on their active and reactive power sharing properties.

I. INTRODUCTION

Microgrids have been envisioned as one of the leading technologies to increase the penetration of renewable energies in the power market. A thorough discussion of the technological, physical and control-theoretic aspects of microgrids is provided in many interesting comprehensive works, including [61], [60], [26], [4], [41].

Power electronics allows inverters in the microgrids to emulate desired dynamic behaviour. This is an essential feature since when the microgrid is in grid forming mode, inverters have to inject active and reactive power in order to supply the loads in a shared manner and maintain the desired frequency and voltage values at the nodes. Hence, much work has focused on the design of dynamics for the inverters that achieve these desired properties, and this effort has involved both practitioners and theorists, all providing a myriad of solutions, whose performance has been tested mainly numerically and experimentally.

The main obstacle however remains a systematic design of the microgrid controllers that achieve the desired properties in terms of frequency and voltage regulation with power sharing. The difficulty lies in the complex structure of these systems, comprising dynamical models of inverters and loads that are physically interconnected via exchange of active and reactive power. In quasi steady state working conditions, these quantities are sinusoidal terms depending on the voltage phasor relative phases. As a result, mathematical models of microgrids reduce to high-order oscillators interconnected via sinusoidal coupling, where the coupling weights depend on the voltage magnitudes obeying additional dynamics. The challenges with these models originates from the presence of highly nonlinear terms and the strict coupling between active and reactive power flow equations.

To deal with the aforementioned complexity of these dynamical models common remedies are to decouple frequency and voltage dynamics, and to linearize the power flow equations. While the former enables a separate analysis of the two dynamics [43], the latter permits the use of a small signal argument to infer stability results; see e.g. [47], [48].

Results that deal with the fully coupled system are also available [42], [55], [36]. In this case, the results mainly concern network-reduced models with primary control, namely stability rather than stabilization of the equilibrium solution. Furthermore, lossy transmission lines can also be studied [21], [55], [6], [55], [36], as well as [15].

Main contribution. In spite of these many advances, what is still missing is a comprehensive approach to deal with the analysis and control design for microgrids. In this paper we provide a contribution in this direction. The starting point is the energy function associated with the system, a combination of kinetic and potential energy. Relying on an extended notion of incremental dissipativity, a number of so-called Bregman storage functions whose critical points have desired features are constructed. The Bregman storage function is a metrics measuring the distance of the actual solution of the microgrid dynamics from a desired one. Its special form allows the designer to unveil an incremental passivity property of the microgrid dynamics, which is then used to design a feedback controller. The construction of the Bregman functions is inspired by works in the control of network systems in the presence of disturbances, which makes use of internal model controllers [9], [35] and incremental passivity [50]. The storage functions that we design encompass several network-reduced versions of microgrid dynamics that have appeared in the literature, including the conventional droop controller [61], [42], the quadratic droop controller [47], and the reactive power consensus dynamics [43]. Our analysis, however, suggests suitable modifications such as an exponential scaling of the averaging reactive power dynamics of [43], and inspires new controllers, such as the so-called reactive current controller (we refer to [7] for a related controller).

The approach we propose has two additional distinguishing features: we do not need to assume decoupled dynamics and we perform a large signal analysis.

Our contribution also expands the knowledge on the use of energy functions in the context of microgrids. Although historically energy functions have played a crucial role to cope
with accurate models of power systems [53], [16], [14], our approach based on the incremental dissipativity notion sheds a new light into the construction of these energy functions, allows us to cover a wider range of microgrid dynamics, and paves the way for the design of dynamic controllers, following the combination of passivity techniques and internal model principles as in [9]. We refer the reader to e.g. [37], [20] for seminal work on passivity-based control of power networks.

In this paper we focus on network reduced models of microgrids [42], [55], [36], [48]. These models are typically criticized for not providing an explicit characterization of the loads [47]. Focusing on network reduced models allows us to reduce the technical complexity of the arguments and to provide an elegant analysis. However, one of the advantages of the use of the energy functions is that they remain effective also with network preserved models [53]. In fact, a preliminary investigation not reported in this manuscript for the sake of brevity shows that the presented results extend to the case of network preserved models [53]. In fact, a preliminary investigation not reported in this manuscript for the sake of brevity shows that the presented results extend to the case of network preserved models [53]. In fact, a preliminary investigation not reported in this manuscript for the sake of brevity shows that the presented results extend to the case of network preserved models [53].

The vector $\theta_n$ of voltage magnitudes. The integer $n$ is the frequency, associated with the nodes. The matrices $T, Q \in \mathbb{R}^{n}$ are the active power vector, $Q \in \mathbb{R}^{n}$ is the reactive power vector, and $V \in \mathbb{R}^{n}$ is the vector of voltage magnitudes. The integer $n$ equals the number of nodes in the microgrid and $T := \{1, 2, \ldots, n\}$ is the set of indices associated with the nodes. The matrices $T_P, T_V, K_P$ are diagonal and positive definite. The vectors $\omega^*$, $P^*$ denote the frequency and active power setpoints, respectively, with $\omega^*$ having all the entries equal to the nominal frequency (e.g., 50 or 60 Hz).

The vector $P^*$ may also model active power loads at the buses (see Remark 2). The vector $u_Q$ is an additional input.

The function $f$ accounts for the voltage dynamics/controller and is decided later. At each node, the model (1) represents the dynamics of the invert at that node in closed-loop with a frequency and voltage controller.

The model (1) with an appropriate selection of $f$ describes various models of network-reduced microgrids in the literature, including conventional droop controllers, quadratic droop controllers, and consensus based reactive power control schemes [61], [46], [42], [47], [43]. However, while [46], [47], [44] consider network-preserved models of microgrids, in this paper network-reduced models are considered. We refer the reader to [44] for a compelling derivation of microgrid models from first principles.

Our goal here is to provide a unifying framework for analysis of the microgrid model (1) for different types of voltage controllers, and to study frequency regulation, voltage stability, and active as well as reactive power sharing. A key point of our approach is that it does not rely on simplifying and often restrictive premises such as the decoupling assumption and linear approximations.

Active and reactive power. The active power $P_i$ is given by

$$P_i = \sum_{j \in N_i} B_{ij} V_i V_j \sin \theta_{ij}, \quad \theta_{ij} := \theta_i - \theta_j$$

and the reactive power by

$$Q_i = B_{ii} V_i^2 - \sum_{j \in N_i} B_{ij} V_i V_j \cos \theta_{ij}, \quad \theta_{ij} := \theta_i - \theta_j.$$  

Note that here $N_i$ is the set of nodes connected to inverter $i, B_{ii} = B_{ii} + \sum_{j \in N_i} B_{ij}$, where $B_{ij} = B_{ji} > 0$ is the negative of the susceptance at edge $(i, j)$ and $B_{ii} \geq 0$ is the negative of the shunt susceptance at node $i$. Hence, $B_{ii} \geq \sum_{j \in N_i} B_{ij}$ for all $i$.

It is useful to have compact representations of both active and reactive power. Setting $\Gamma(V) = \text{diag}(\gamma_1(V), \ldots, \gamma_m(V))$, $\gamma_k(V) = V V_j B_{ij}$, with $k \in E := \{1, 2, \ldots, m\}$ being the index corresponding to the edge $(i, j)$ (in short, $k \sim \{i, j\}$), the vector of the active power at all the nodes is written as

$$P = D \Gamma(V) \sin(D^T \theta).$$

where $D = [d_{ik}]$ is the incidence matrix of the graph describing the interconnection structure of the network, and the vector $\sin(\cdot)$ is defined element-wise. Let us now introduce the vector $A_0 = \text{col}(B_{11}, \ldots, B_{mn})$. Since $|d_{ik}| \cos(d_{ik} \theta_i + d_{jk} \theta_j) = \cos(\theta_i - \theta_j)$, for $k \sim \{i, j\}$, the vector of reactive power at the nodes takes the form

$$Q = [V] |A_0| V - [D] \Gamma(V) \cos(D^T \theta),$$

where $|D|$ is obtained by replacing each element $d_{ij}$ of $D$ with

\[^1\text{See Remark 2 for a discussion on the physical meaning of these shunt susceptances.}\]
Moreover, here and throughout the paper, the notation $[v]$ represents the diagonal matrix associated with vector $v$.

Another compact representation is useful as well. To this end, introduce the symmetric matrix
\[
A(\cos(D^T \theta)) = \begin{bmatrix}
B_{11} & \cdots & -B_{1n} \\
-\cos \theta_{12} & \ddots & \cdots \\
\vdots & \ddots & \ddots \\
-B_{n1} & \cdots & B_{nn} 
\end{bmatrix},
\]

where again we are exploiting the identity $\cos(d_{ij} \theta_i + d_{jk} \theta_j) = \cos \theta_{ij}$.

As a consequence of the condition $B_{ii} \geq \sum_{j \in \mathcal{N}} B_{ij}$ for all $i$, provided that at least one $B_{ii}$ is non-zero (which is the standing assumption throughout the paper), the symmetric matrix $A(\cos(D^T \theta))$ has all strictly positive eigenvalues and hence is a positive definite matrix. Note that the matrix $A$ can be interpreted as a loopy Laplacian matrix of the graph.

Before proceeding further, we remark on the adopted model.

**Remark 1. (Lossless and lossy network)** The power lines are assumed to be lossless in (1). This is valid if the lines are dominantly inductive, a condition which can be fulfilled by tuning output impedances of the inverters; see e.g. [32]. As will be observed in Subsection VII-A, the lossless assumption can be relaxed by considering lossy, yet homogenous, power lines.

**Remark 2. (Loads)** There are a few load scenarios that can be incorporated in the microgrid model (1). The first scenario accounts for purely inductive loads, see [43, Remark 1]. Whether these loads are collocated with inverters or appear as individual nodes, they will lead to nonzero shunt admittances at the nodes of the reduced network, where the latter follows from Kron reduction. The resulting shunt admittances constitute the nonzero shunt susceptance $B_{ii}$ introduced after (3), see also [48, Section V.A] and [43]. As for the active power loads, following [42, Remark 3.2], one can consider negative active power setpoints $P_i^*$ for the inverter $i$, which corresponds to the inverter $i$ connecting a storage device to the grid, in which case the device is acting as a frequency and voltage dependent load; see also [36, Section 2.4]. Another possibility is to consider constant active power loads collocated with the inverters by embedding the constant active power consumption in the term $P_i^*$. We remark that the controllers studied in the paper do not rely on the knowledge of $P_i^*$, and are therefore fully compatible with the case in which $P_i^*$ are not completely known due to uncertainties in the loads. Finally, the extension of our analysis to the lossy lines in Subsection VII-A allows us to accommodate loads as homogenous RL circuits. As an interesting special case of this, the forthcoming dissipativity/stability analysis carries over to the case of microgrids with (purely) resistive lines and loads. More details on this case are provided in Subsection VII-A.

To pursue our analysis, we demonstrate an incremental cyclo-dissipativity property of the various microgrid models, with respect to a “synchronous solution”. The notion of dissipativity adopted in this paper is introduced next, and synchronous solutions will be identified afterwards.

**Definition 1.** System $\dot{x} = f(x,u), y = h(x)$, with states $x \in \mathcal{X}$, input and output signals $u, y \in \mathbb{R}^m$, is incrementally cyclo-dissipative with state-dependent supply rate $s(x,u,y)$ and with respect to a given input-state-output triple $(\overline{x}, \overline{u}, \overline{y})$, if there exist a continuously differentiable function $S: \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, and state-dependent positive semi-definite3 matrices $W, R: \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^m$, such that for all $x \in \mathcal{X}, u \in \mathbb{R}^m$ and $y = h(x)$, $\kappa = (\overline{x}, \overline{u}, \overline{y})$, the inequality
\[
\frac{\partial S}{\partial x}(x,u) + \frac{\partial S}{\partial \kappa}\left(\overline{x}, \overline{u}, \overline{y}\right) \leq s(x,u-\overline{u}, y-\overline{y})
\]
holds with
\[
s(x,u,y) = -y^T W(x)y + y^T R(x)u.
\]

We remark that at this point the function $S$ is not required to be non-negative nor bounded from below, and that the weight matrices $W, R$ are allowed to be state dependent. The use of the qualifier “cyclo” in the definition above stresses the former feature [54, Def. 2]. This qualifier can be dropped once conditions guaranteeing the positive definiteness of the storage function are in place (see Proposition 1).

**Remark 3.** In case the matrices $W$ and $R$ are state independent, some notable special cases of Definition 1 are obtained as follows:

i) $W \geq 0, R = I, S \geq 0$ (incremental passivity)

ii) $W > 0, R = I, S \geq 0$ (output-strict incremental passivity)

iii) $W \geq 0, R = I$ (cyclo-incremental passivity)

iv) $W > 0, R = I$ (output-strict cyclo-incremental passivity).

**Synchronous solution.** Given the constant vectors $\overline{u}_i, \overline{u}_q$, a synchronous solution to (1) is defined as the triple
\[
(\theta(t), \omega(t), V(t)) = (\overline{\theta}, \overline{\psi}, \overline{V}),
\]

3A state-dependent matrix $M : \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^m$ is positive semi-definite if $y^T M(x)y \geq 0$ for all $x \in \mathcal{X}$ and for all $y \in \mathbb{R}^m$. If $M$ is positive semi-definite and $y^T M(x)y = 0 \iff y = 0$ then $M$ is called positive definite.

4We are slightly abusing the classical notion of incremental dissipativity [19], for we do not consider pairs of arbitrary input-state-output triples, but pairs in which one of the two triples is fixed. For additional work on incremental dissipativity we refer the reader to [49], [50].
where $\theta = \omega t + \theta^0$, $\omega = 1 \omega^0$, the scalar $\omega^0$ and the vectors $\theta^0$ and $\nabla \in \mathbb{R}^n_0$ are constant. In addition,

$$
0 = -(\bar{\omega} - \omega^0) - K_P(\bar{P} - P^* + \bar{\bar{u}}), \quad 0 = f(\bar{V}, \bar{Q}, \bar{u}_Q),
$$

(6)

where

$$
\bar{P} = D \Gamma(\nabla) \sin(D^T \bar{\theta}), \quad \bar{Q} = |\nabla| A(\cos(D^T \bar{\theta}) \nabla = |\nabla| A(\cos(D^T \theta)) \nabla.
$$

(7)

In (6) and the rest of the paper, the symbol $\theta$ denotes a vector or a matrix of appropriate dimension with entries all equal to zero. Notice that the key feature of a synchronous solution is that the voltage phase angles are rotating with the same frequency, namely $\omega^0$, and the differences of these angles are thus constant. Another feature is that the voltage amplitudes are constant.

III. DESIGN OF BREGMAN STORAGE FUNCTIONS

A crucial step for the Lyapunov based analysis of the coupled nonlinear model (1) is constructing a storage function. Inspired by the classical power system stability analysis, see e.g. [39], we start off with the following energy-based function

$$
U(\theta, \omega, V) = \frac{1}{2} \omega^T K_P^{-1} T_P \omega + \frac{1}{2} V^T Q,
$$

(8)

where we have exploited (4) to write the second equality. Notice that the first term represents the kinetic “energy” (in quotes because the term has the units of power and it does not correspond to the physical inertia), and the second term represent the sum of the reactive power stored in the links and the power partly associated with the shunt components.

Next, we compute the gradient of the storage function as follows:

$$
\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial V} = \frac{\partial U}{\partial V} = \frac{\partial U}{\partial V} = [V]^{-1} Q = [A_0] V - [V]^{-1} [D] \Gamma(V) \cos(D^T \theta).
$$

In the latter equality, we are implicitly assuming that each component of the voltage vector never crosses zero. In fact, we shall assume the following:

**Assumption 1.** There exists a subset $\mathcal{X}$ of the state space $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n_0$ that is forward invariant along the solutions to (1).

Conditions under which this assumption is fulfilled will be provided later in the paper.

Notice that the voltage dynamics identified by $f$ have not yet been taken into account in the function $U$. Therefore, to cope with different voltage dynamics (or controllers) we add another component, namely $H(V)$, and define

$$
S(\theta, \omega, V) = U(\theta, \omega, V) + H(V).
$$

(9)

We rest our analysis on the following foundational incremental storage function

$$
S(\theta, \omega, V) = S(\theta, \omega, V) - S(\bar{\theta}, \bar{\omega}, \bar{V}) - \frac{\partial S}{\partial \theta}^T (\theta - \bar{\theta}) - \frac{\partial S}{\partial \omega}^T (\omega - \bar{\omega}) - \frac{\partial S}{\partial V}^T (V - \bar{V})
$$

(10)

where we use the conventional notation

$$
\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} (\pi), \quad \frac{\partial F}{\partial x}^T = \left( \frac{\partial F}{\partial x} (\pi) \right)^T.
$$

(11)

for a function $F : \mathcal{X} \rightarrow \mathbb{R}$. The storage function $S$, in fact, defines a “distance” between the value of $S$ at a point $(\theta, \omega, V)$ and the value of a first-order Taylor expansion of $S$ around $(\bar{\theta}, \bar{\omega}, \bar{V})$. This construction is referred to as **Bregman distance** or **Bregman divergence** following [8], and has found its applications in convex programming, clustering, proximal minimization, online learning, and proportional-integral stabilisation of nonlinear circuits; see e.g. [8], [3], [12], [56], [29]. In thermodynamics, the Bregman distance has its antecedents in the notion of availability function [31], [1], [57].

The function $S$ can be decomposed as

$$
S = U + H
$$

(12)

where

$$
U(\theta, \omega, V) = U(\theta, \omega, V) - U(\bar{\theta}, \bar{\omega}, \bar{V}) - \frac{\partial U}{\partial \theta}^T (\theta - \bar{\theta}) - \frac{\partial U}{\partial \omega}^T (\omega - \bar{\omega}) - \frac{\partial U}{\partial V}^T (V - \bar{V})
$$

and

$$
H(V) = H(V) - H(\bar{V}) - \frac{\partial H}{\partial V}^T (V - \bar{V}).
$$

(13)

The above identities show that the critical points of $S$ occur for $\omega = \bar{\omega}$ and $P = \bar{T}$, which is a desired property. The critical point of $S$ with respect to the $V$ coordinate is determined by the choice of $H$, which depends on the voltage dynamics.

To establish a suitable incremental dissipativity property of the system (1) with respect to a synchronous solution, we introduce the output variables

$$
y = \text{col}(y_P, y_Q)
$$

(12)

with

$$
y_P = T_P^{-1} \frac{\partial S}{\partial \omega} = K_P^{-1} \omega, \quad y_Q = T_Q^{-1} \frac{\partial S}{\partial V},
$$

and input variables

$$
u = \text{col}(u_P, u_Q).
$$

(13)

In what follows, we discriminate among different voltage controllers and adjust the analysis accordingly by tuning $H$. 
A. Conventional droop controller

The conventional droop controllers are obtained by setting $f$ in (1) as

$$f(V, Q, u_Q) = -V - K_Q Q + u_Q$$

(14)

where $K_Q = [k_Q]$ is a diagonal matrix with positive droop coefficients on its diagonal. Note that $u_Q$ is added for the sake of generality and one can set $u_Q = \pi_Q = K_Q Q^* + V^*$ for nominal constant vectors $V^*$ and $Q^*$ to obtain the well known expression of conventional droop controllers, see e.g. [13], [61]. For this choice of $f$, we pick the function $H$ in (9) as [42], [51]

$$H(V) = 1^T K_Q V - (\bar{Q} + K_Q^{-1} \nabla)^T \ln(V),$$

(15)

with $\bar{Q} + K_Q^{-1} \nabla = K_Q^{-1} \pi_Q \in \mathbb{R}^n_>$ and $\ln(V)$ defined element-wise. The choice of $H$ in (15) has two interesting features. First, it makes the incremental storage function $S$ radially unbounded with respect to $V$ on the positive orthant. Moreover, it shifts the critical points of $S$ as desired. Noting that by (6)

$$\bar{0} = -\nabla - K_Q \bar{Q} + \pi_Q,$$

a straightforward calculation yields

$$T_Q \dot{V} = -K_Q [V] \frac{\partial S}{\partial V} + u_Q - \pi_Q.$$  

(16)

In the following subsections we will derive analogous identities and then use those for concluding incremental cyclo-dissipativity of the system.

B. Quadratic droop controller

Another voltage dynamics proposed in the literature are associated with the quadratic droop controllers of [47], which can be expressed as (1) with

$$f(V, Q, u_Q) = -K_Q Q - [V](V - u_Q),$$

(17)

where again $K_Q = [k_Q]$ collects the droop coefficients. The quadratic droop controllers in [47] are obtained by setting $u_Q = V^*$ for some constant vector $V^*$. Notice however the difference: while [47] focuses on a network preserved microgrid model in which the inverter dynamics are decoupled from the frequency dynamics, here a fully coupled network reduced model is considered.

Moreover, note that the scaling matrix $[V]$ distinguishes the quadratic droop controller from the conventional one. For the controller (17), we change the storage function $S$ by setting

$$H(V) = \frac{1}{2} V^T K_Q^{-1} V.$$  

(18)

Recall that $S = \mathcal{U} + \mathcal{H}$. Note that $S$ is defined on the whole space $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and not on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n$. The resulting function $S$ can be interpreted as a performance criterion in a similar vein as the cost function in [47]. Noting that

$$\bar{0} = -K_Q \bar{Q} - [V](V - \bar{Q}),$$

it is easy to verify that

$$T_Q \dot{V} = -K_Q [V] \frac{\partial S}{\partial V} + [V](u_Q - \pi_Q).$$  

(19)

C. Reactive current controller

The frequency dynamics of the inverters in microgrids typically mimic that of the synchronous generators known as the swing equation. This facilitates the interface of inverters and generators in the grid. To enhance such interface further, one can mimic the voltage dynamics of the synchronous generators as well. Motivated by this, we consider the voltage dynamics

$$f(V, Q, u_Q) = -[V]^{-1} Q + u_Q.$$  

(20)

This controller aims at regulating the ratio of reactive power over voltage amplitudes, which can be interpreted as “reactive current” [33]. For this controller, we set

$$H = 0,$$  

(21)

meaning that $S = U$ and no adaptation of the storage function is needed. It is easy to observe that

$$T_Q \dot{V} = -\frac{\partial S}{\partial V} + u_Q - \bar{u}_Q,$$  

(22)

where $\bar{u}_Q = [V]^{-1} \bar{Q}$ is again the feedforward input guaranteeing the preservation of the steady state.

D. Exponentially-scaled averaging reactive power controller

In this subsection, we consider another controller that tries to achieve proportional reactive power sharing

$$f(V, Q, u_Q) = -[V] K_Q L_Q K_Q Q + [V] u_Q$$  

(23)

where $K_Q = [k_Q]$ is a diagonal matrix, and $L_Q$ is the Laplacian matrix of a connected and undirected communication graph. Compared with the controller in [43], here the voltage dynamics are scaled by the voltages at the inverters, namely $[V]$, the reactive power $Q$ is not assumed to be independent of the phase variables $\theta$, and an additional input $u_Q$ is introduced. It is easy to see that the voltage dynamics in this case can be equivalently rewritten as

$$T_Q \dot{\chi} = -K_Q L_Q K_Q Q + u_Q$$  

(24)

$$V = \exp(\chi)$$  

where $\chi$ can be expressed in terms of $\chi$ as $[\exp(\chi)]_A(\cos(D^T \theta)) \exp(\chi)$ with $\exp(\chi) = \text{col}(e^{\chi_i})$. Hence, we refer to this controller as an exponentially-scaled averaging reactive power controller (E-ARP). Now, we choose $\dot{H}$ as

$$H(V) = -\bar{Q}^T \ln(V),$$  

(25)

with $\bar{Q}$ as in (7), and obtain

$$\frac{\partial S}{\partial V} = [V]^{-1} (Q - \bar{Q}).$$  

(26)

Note that, in fact, our treatment here together with the equality (26) above hints at the inclusion of the matrix $[V]$ into the controller, or equivalently at an exponential scaling of the reactive power averaging dynamics (see (23), (24)). This, as will be observed, results in reactive power sharing for the fully coupled nonlinear model (1). By defining

$$\pi_Q = K_Q L_Q K_Q \bar{Q},$$  

(27)
the voltage dynamics can be rewritten as
\[ \dot{V} = -[V]K_Q L_Q K_Q[V] \frac{\partial S}{\partial V} + [V](u_Q - \pi_Q). \] (28)
where we have set \( T_Q = I \). Requiring the time constants to be identical to each other is a purely technical assumption, motivated by the difficulty of analysing the system without this uniformity condition. The assumption of unitary time constants, on the other hand, is made only for the sake of simplicity, and can be easily relaxed.

IV. INCREMENTAL CYCLO-DISSIPATIVITY OF MICROGRID MODELS

In this section, we show how the candidate Bregman storage functions we introduced before allow us to infer incremental cyclo-dissipativity of the microgrids under the various controllers.

**Theorem 1.** Assume that the feasibility condition (6) admits a solution and let Assumption 1 hold. Then system (1) with output (12), input (13), and, respectively,

1) \( f(V, Q, u_Q) \) given by (14);
2) \( f(V, Q, u_Q) \) given by (17);
3) \( f(V, Q, u_Q) \) given by (20);
4) \( f(V, Q, u_Q) \) given by (23);

is incrementally cyclo-dissipative with respect to a synchronous solution \((\bar{\theta}, \bar{\omega}, \bar{V})\), with

1) incremental storage function \( S \) defined by (8),(9),(10) and (15) and supply rate (5) with weight matrices
\[ W(V) = \begin{bmatrix} K_P & 0 & 0 \\ T_Q K_Q[V] & 0 & 0 \end{bmatrix} \]
\[ R = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} ; \]

2) incremental storage function \( S \) defined by (8),(9),(10),(18) and supply rate (5) with weight matrices
\[ W(V) = \begin{bmatrix} K_P & 0 \\ 0 & T_Q K_Q[V] \end{bmatrix} ; \]
\[ R(V) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} ; \]

3) incremental storage function \( S \) defined by (8),(9),(10),(21) and supply rate (5) with weight matrices
\[ W(V) = \begin{bmatrix} K_P & 0 \\ 0 & T_Q \end{bmatrix} ; \]
\[ R = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} ; \]

4) incremental storage function \( S \) defined by (8),(9),(10),(25) and supply rate (5) with weight matrices
\[ W(V) = \begin{bmatrix} K_P & 0 \\ 0 & T_Q K_Q[V] \end{bmatrix} ; \]
\[ R(V) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} . \]

**Proof:** 1) Recall the equalities
\[ \frac{\partial S}{\partial \omega} = K_P^{-1} T_P(\omega - \bar{\omega}), \]
\[ \frac{\partial S}{\partial \bar{\omega}} = D \Gamma(V) \sin(D^T \theta) - D \Gamma(\bar{V}) \sin(D^T \theta^0) = (P - \bar{P}). \]
Then
\[ \frac{d}{dt} S = (\omega - \bar{\omega})^T T_P K_P^{-1} \dot{\omega} \]
\[ + (D \Gamma(V) \sin(D^T \theta) - D \Gamma(\bar{V}) \sin(D^T \theta^0))^T \dot{\theta} + (\frac{\partial S}{\partial \omega})^T \omega \]
\[ = (\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega} - K_P(P - \bar{P}) + (u_P - \pi_P)) \]
\[ + (D \Gamma(V) \sin(D^T \theta) - D \Gamma(\bar{V}) \sin(D^T \theta^0))^T (\omega - \bar{\omega}) \]
\[ + \frac{\partial S}{\partial \omega}^T T_Q^{-1} [K_Q[V] \frac{\partial S}{\partial V} + (\omega - \bar{\omega})^T T_Q^{-1} K_Q[V] \frac{\partial S}{\partial V} + (\frac{\partial S}{\partial \omega})^T T_Q^{-1} (u_Q - \bar{u_Q})] \]
where the chain of equalities hold because of the feasibility condition and (16). Hence
\[ \frac{d}{dt} S = -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \pi_P) \]
\[ - (\frac{\partial S}{\partial \omega})^T T_Q^{-1} K_Q[V] \frac{\partial S}{\partial V} + (\frac{\partial S}{\partial \omega})^T T_Q^{-1} (u_Q - \bar{u_Q}) \]
\[ (29) \]
Observe now that by definition
\[ \frac{\partial S}{\partial \omega} = \frac{\partial S}{\partial V} - \frac{\partial S}{\partial \bar{V}} \]
and that \( \frac{\partial S}{\partial \bar{V}} \) represents the output component \( \frac{\partial S}{\partial \bar{V}} \) at a synchronous solution. Hence equality (31) at the top of the next page can be established.

We conclude incremental cyclo-dissipativity of system (1), (12), (13), (14) as claimed.

2) If, in the chain of equalities defining \( \frac{d}{dt} S \) above, we use (19) instead of (16), we obtain that
\[ \frac{d}{dt} S = -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \pi_P) \]
\[ - (\frac{\partial S}{\partial \omega})^T T_Q^{-1} K_Q[V] \frac{\partial S}{\partial V} + (\frac{\partial S}{\partial \omega})^T T_Q^{-1} (u_Q - \bar{u_Q}), \]
\[ (31) \]
which shows incremental cyclo-dissipativity of system (1), (12), (13), (17).

3) For this case, adopting the equality (22) results in the equality
\[ \frac{d}{dt} S = -(\omega - \bar{\omega})^T K_P^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_P^{-1} (u_P - \pi_P) \]
\[ - (\frac{\partial S}{\partial \omega})^T T_Q^{-1} \frac{\partial S}{\partial V} + (\frac{\partial S}{\partial \omega})^T T_Q^{-1} (u_Q - \bar{u_Q}), \]
\[ (32) \]
from which incremental cyclo-dissipativity of (1), (12), (13), (20) holds.
4) Finally, in view of (28), we have
\[
\frac{d}{dt} S = -((\omega - \bar{\omega})^T K_p^{-1} (\omega - \bar{\omega}) + (\omega - \bar{\omega})^T K_p^{-1} (u_p - \bar{u}_p))
- (\frac{\partial S}{\partial V})^{T}[V] K_Q L_Q K_Q [V] \frac{\partial S}{\partial V} + (\frac{\partial S}{\partial V})^{T}[V](u_Q - \bar{u}_Q),
\]
which implies incremental cyclo-dissipativity of (1), (12), (13), (23).

V. STABLE AND UNSTABLE EQUILIBRIA

The dissipation inequalities proven before can be exploited to study the stability of a synchronous solution. Recall that Theorem 1 has been established in terms of cyclo-dissipativity rather than dissipativity, i.e., without requiring the storage function $S$ to be bounded from below. To investigate the attractivity of a synchronous solution, we examine the shape of the storage functions. To this end, first we map synchronous solutions to equilibrium points of the system by a suitable change of coordinates. Then, we study the convexity of the storage function at an equilibrium. To complement this result, we propose sufficient conditions for instability of equilibria of the micogrid.

A. Change of coordinates

It is not difficult to observe that due to the rotational invariance of the $\theta$ variable, the existence of a strict minimum for $S$ cannot be anticipated. To clear this obstacle, we notice that the phase angles $\theta$ appear as relative terms, i.e., $D^T \theta$, in (8) and thus in $S$ as well as $\partial S$. Motivated by this observation, we introduce the new variables [58]
\[
\varphi_i = \theta_i - \theta_n, \quad i = 1, 2, \ldots, n - 1.
\]
These can be also written as
\[
\begin{bmatrix}
\varphi_1 \\
\vdots \\
\varphi_{n-1} \\
\end{bmatrix}
= \begin{bmatrix}
\theta_1 \\
\vdots \\
\theta_{n-1} \\
\end{bmatrix} - \mathbf{1} \theta_n.
\]
Let us partition $D$ accordingly as $D = \text{col}(D_1, D_2)$, with $D_1$ an $(n - 1) \times m$ matrix and $D_2$ a $1 \times m$ matrix. Notice that $D_1$ is the reduced incidence matrix corresponding to the node of index $n$ taken as reference. Then
\[
D^T \begin{bmatrix}
\varphi_1 \\
\vdots \\
\varphi_{n-1} \\
0
\end{bmatrix}
= D_1^T \varphi, \quad \text{with} \quad \varphi := \begin{bmatrix}
\varphi_1 \\
\vdots \\
\varphi_{n-1} \\
\end{bmatrix},
\]
and therefore
\[
D^T \theta = D_1^T \varphi.
\]

More explicitly, given $\theta \in \mathbb{R}^n$, we can define $\varphi \in \mathbb{R}^{n-1}$ from $\theta$ as in (34), and the equality $D^T \theta = D_1^T \varphi$ holds. Hence,
\[
U(\theta, \omega, V) = \frac{1}{2} \omega^T K_p^{-1} T_p \omega + \frac{1}{2} V^T A(\cos(D^T \theta)) V
= \frac{1}{2} \omega^T K_p^{-1} T_p \omega + \frac{1}{2} V^T A(\cos(D_1^T \varphi)) V := \dot{U}(\varphi, \omega, V).
\]

To ease the notation, in what follows, we drop the hat on $U(\varphi, \omega, V)$. Then, we can define
\[
\begin{align*}
\bar{U}(\varphi, \omega, V) &= U(\varphi, \omega, V) - U(\bar{\varphi}, \bar{\omega}, \bar{V}) \\
- \frac{\partial U}{\partial \varphi}^T (\varphi - \bar{\varphi}) - \frac{\partial U}{\partial \omega}^T (\omega - \bar{\omega}) - \frac{\partial U}{\partial V}^T (V - \bar{V})
\end{align*}
\]
where, $\bar{\varphi}_i := \bar{\theta}_i - \bar{\theta}_n, \; i = 1, 2, \ldots, n - 1$, (hence $D^T \bar{\theta} = D_1^T \bar{\varphi}$), and
\[
S(\varphi, \omega, V) = \bar{U}(\varphi, \omega, V) + H(V)
\]
to have
\[
S(\theta, \omega, V) \equiv S(\varphi, \omega, V).
\]

B. Stable equilibria

Observe that $(\bar{\varphi}, \bar{\omega}, \bar{V})$ is a critical point of $S(\varphi, \omega, V)$. Next, we compute the Hessian as
\[
\frac{\partial^2 S}{\partial (\varphi, \omega, V)^T} =
\begin{bmatrix}
D_1 \Gamma[V](\cos(D_1^T \varphi)) & D_1^T & 0 \\
D_1[V]^{-1} & D_1[V](\sin(D_1^T \varphi)) & 0 & A(\cos(D_1^T \varphi)) + \frac{\partial^2 H}{\partial \varphi^2}
\end{bmatrix}.
\]

Notice that in all the previously studied cases, the matrix $\frac{\partial^2 H}{\partial \varphi^2}$ is diagonal. In particular,
\[
\begin{align*}
\frac{\partial^2 H}{\partial \varphi^2} &= K_Q + [V]^{-2}[Q + K_Q^{-1} V], \quad \frac{\partial^2 H}{\partial \omega^2} = K_Q^{-1}, \\
\frac{\partial^2 H}{\partial V^2} &= 0, \quad \frac{\partial^2 H}{\partial \varphi^2} = [V]^{-2}[Q],
\end{align*}
\]
for conventional droop, quadratic droop, reactive current controller, and exponentially-scaled averaging reactive power controller, respectively from left to right. Now, let
\[
[h(V)] := \frac{\partial^2 H}{\partial \varphi^2},
\]
and $h(V) = \text{col}(h_i(V_i))$. Then, we can prove the following result, which establishes distributed conditions for checking the positive definiteness of the Hessian, and hence strict convexity of the Bregman storage function.
Proposition 1. Let \( \eta := D^T \theta^0 = D_1^T \varphi \in (\eta_1, \eta_2)^m, \varphi \in R^n_+, \) and

\[
m_{ii} := \dot{B}_{ii} + \sum_{k \sim (i, t) \in E} B_{it} \left(1 - \frac{\nabla_i \sin^2(\eta_k)}{\cos(\eta_k)}\right) + h_i(\nabla_i).
\]

Then the inequality

\[
\frac{\partial^2 S}{\partial (\varphi, \omega, V)^2} > 0
\]

holds if

\[
m_{ii} > \sum_{k \sim (i, t) \in E} B_{it} \sec(\eta_k)
\]

for all \( i = 1, 2, \ldots, n. \)

Proof: The proof is given in the appendix.

Remark 4. (Stable equilibria) By (42), the conditions for positive definiteness are met provided that at the point \((\varphi, \omega, V)\) the relative voltage phase angles are small enough and the voltages magnitudes are sufficient uniform. This is an interesting condition stating that if the equilibria of interest are characterized by small relative voltage phases and comparable voltage magnitudes, then they are isolated minima of \(S(\varphi, \omega, V).\) In view of Theorem 1, these points are in fact stable equilibria of the microgrid (1), expressed in the new coordinates, with \(u_P = \pi_P\) and \(u_Q = \pi_Q;\) see also Remark 8. A more detailed characterization of the set of stable equilibria is left for future research.

Remark 5. (Hessian) The Hessian of energy functions has always played an important role in stability studies of power networks; see e.g. [53], and [42] for a microgrid stability investigation. Conditions for assessing the positive definiteness of the Hessian of an energy function associated to power networks have been reported in the literature since [53], and used even recently to study e.g. the convexity of the energy function [25]. Our conditions however are different and hold for more general energy functions.

C. Unstable equilibria

Conversely, one can characterize an instability condition that shows how, for a given vector of voltage values, equilibria with “large” relative phase angles are unstable. To this end, first observe that a negative eigenvalue of the Hessian matrix implies instability of the equilibrium \((\varphi, \omega, \varphi)\) of system (1), with \(f(V, Q, u_Q)\) given by (14), (17), (20), expressed in the \(\varphi\) coordinates and with \(u_P = \pi_P, u_Q = \pi_Q:\)

Lemma 1. Suppose that the Hessian

\[
\frac{\partial^2 S}{\partial (\varphi, \omega, V)^2} > 0
\]

is nonsingular and has a negative eigenvalue. Then the equilibrium \((\varphi, \omega, V)\) is unstable.

Proof: The proof is omitted for lack of space and can be found in [63].

Before providing sufficient conditions under which the Hessian in Lemma 1 has a negative eigenvalue, we first provide conditions under which the matrix at the center of the product in (44), denoted as \(M\) when evaluated at \((\varphi, \omega, \varphi)\), has a negative eigenvalue. \(M\) is symmetric and as such diagonalizable. Using the diagonal form, it is immediate to notice that if there exists a vector \(v = (v(1), v(2)) \neq 0\) such that \(v^T M v < 0\), then the matrix \(M\) has a negative eigenvalue.

A characterization of the condition \(v^T M v < 0\), or equivalently the existence of a negative eigenvalue of the matrix \(M\), is now studied. To this end, it is instrumental to introduce a class of cut-sets of the graph, as in the following definition:

Definition 2. A cut-set \(K \subseteq E\) is said to have non-incident edges if for each \(k \sim \{i, j\} \in K\) and \(k' \sim \{i', j'\} \in K,\) with \(k \neq k',\) all the indices \(i, j, i', j'\) are different from each other. The set of cut-sets with non-incident edges is denoted by \(K.\)

In words, the property amounts to the following: given any two edges in the cut-set, the two pairs of end-points associated with the two edges are different from each other. The set of graphs for which these cuts exists is not empty and includes trees, rings and lattices. Complete graphs do not admit this class of cuts. Now, the following holds:

Lemma 2. Let \(\varphi \in R^n_+:\) If there exists a cut-set \(K \subseteq K\) such that, for all \(k \sim \{i, j\} \in K,\)

\[
\sin(\eta_k)^2 > \beta_k(\nabla_i, \nabla_j) \cos(\eta_k),
\]

where \(\eta = D_1^T \varphi\) and

\[
\beta_k(\nabla_i, \nabla_j) = 2 \max \{ \frac{(B_{ii} + h_i(\nabla_i)) \nabla_i}{B_{ij} \nabla_j}, \frac{(B_{jj} + h_j(\nabla_j)) \nabla_j}{B_{ij} \nabla_i} \},
\]

and \(h_i\) is defined in (39), (40), then the matrix \(M\) at the center of the product in (44) evaluated at \(\varphi, V\) has a negative eigenvalue.

Proof: A sketch of the proof is provided in [63].

The two lemmas above lead to the following conclusion:

Proposition 2. An equilibrium \((\varphi, \omega, V),\) with \(V \in R^n_+,\) and (43) nonsingular, is unstable if there exists a cut-set \(K \subseteq K\) such that the inequality (45) holds for all \(k \sim \{i, j\} \in K.\)

Proof: The proof is given in the appendix.

From the relation above, we see that for equilibria for which the components of \(V\) have comparable values, inequality (45) is likely to be fulfilled as \(\eta_k\) diverges from 0, thus showing that equilibria with large relative phase angles are likely to be unstable. Related instability conditions based on cuts have appeared in the coupled oscillators literature, see [34].

Remark 6. (Plastic coupling strength) It is interesting to establish a connection with existing studies on oscillator synchronization arising in different contexts. Once again, this connection leverages the use of the energy function. If the coupling between any pair of nodes \(i, j\) is represented by a single variable \(v_{ij},\) modeling e.g. a dynamic coupling, instead of the product of the voltage variables \(V_i V_j,\) then a different model arises. To obtain this, we focus for the sake of simplicity on oscillators without inertia, and replace the previous energy
function (8) with 

\[ U(\theta, v) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} v_{ij} B_{ij} \cos(\theta_j - \theta_i) + \frac{1}{2} \sum_{(i,j) \in E} v_{ij}^2. \]

Then

\[ \frac{\partial U}{\partial v_{ij}} = -B_{ij} \cos(\theta_j - \theta_i) + v_{ij}, \]

and the resulting (gradient) system becomes

\[ \dot{\theta}_i = \sum_{j \in N_i} v_{ij} B_{ij} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, n, \]

\[ \dot{v}_{ij} = B_{ij} \cos(\theta_j - \theta_i) - v_{ij}, \quad (i, j) \in E, \]

which arises in oscillator networks with so-called plastic coupling strength [40], [28], [34] and in the context of flocking with state dependent sensing [40], [24], [45]. Although stability analysis of equilibria have been carried out for these systems, the investigation of the methods proposed in this paper in those contexts is still unexplored and deserves attention.

VI. FREQUENCY CONTROL WITH POWER SHARING

In this section, we establish the attractivity of a synchronous solution, which amounts to the frequency regulation (ω = ω∗) with optimal properties. Moreover, we investigate voltage stability and reactive power sharing in the aforementioned voltage controllers.

A. Attractivity of synchronous solutions

Recall from (6) that for a synchronous solution we have

\[ 0 = -K P(D \Gamma(V) \sin(D^T \theta^0) - P^*) + \pi_P. \quad (46) \]

Among all possible vectors \( \pi_P \) satisfying the equality above, we look for the one that minimizes the quadratic cost function

\[ C(\pi_P) = \frac{1}{2} \pi_P^T K P^{-1} \pi_P. \quad (47) \]

This choice is explicitly computed as [2], [23], [52]

\[ \pi_P = -\frac{1}{2} \frac{1^T P^*}{1^T K P^{-1} 1}. \quad (48) \]

Note that in (47) any positive diagonal matrix, say \( \Sigma \), could be used instead of \( K P^{-1} \). However, the choice \( \Sigma = K P^{-1} \) yields more compact expressions, and results in proportional sharing of the active power according the droop coefficients \( k_{P,i} \), see Subsection VI-B.

Replacing \( \pi_P \) in (6) with its expression in (48), and replacing \( \Gamma \) with its explicit definition via the loopy Laplacian, the feasibility condition (6) can be restated as follows:

**Assumption 2.** There exist constant vectors \( V \in \mathbb{R}^n \) and \( \theta^0 \in \mathbb{R}^n \) such that

\[ D \Gamma(V) \sin(D^T \theta^0) = \left( I - K P^{-1} \frac{1 1^T}{1^T K P^{-1} 1} \right) P^* \quad (49) \]

and

\[ 0 = f(V, |V| A(\cos(D^T \theta^0)))V, \pi_Q). \quad (50) \]

**Remark 7.** Similar to [52, Remark 5] it can be shown that if the assumption above is satisfied then necessarily \( \Gamma(V) \in \mathbb{R}^n_{>0} \).

Furthermore, in case the network is a tree, we know that (49) is satisfied if and only if there exists \( V \in \mathbb{R}^n_{>0} \) such that

\[ ||\Gamma(V)^{-1} D^{\dagger} \left( I - K P^{-1} \frac{1 1^T}{1^T K P^{-1} 1} \right) P^* ||_{\infty} < 1, \]

with \( D^{\dagger} \) denoting the left inverse of \( D \). In the case of the quadratic voltage droop and reactive current controllers, explicit expressions of the voltage vector \( V \) can be given (see Subsection VI-B), in which case the condition above becomes dependent on the voltage phase vector \( \theta^0 \) only.

To achieve the optimal input (48), we consider the following distributed active power controller [46], [23], [10]

\[ \xi = -L P \xi + K P^{-1} (\omega^* - \omega) \]

\[ u_P = \xi \]

where the matrix \( L P \) is the Laplacian matrix of an undirected and connected communication graph. Here, the term \( \omega^* - \omega \) attempts to regulate the frequency to the nominal one, whereas the consensus based algorithm \( -L P \xi \) steers the input to the optimal one given by (48) at steady-state. The controller (51) is distributed because at each node \( i \) only information about the neighbours’ variables \( \xi_j \) and the local frequency deviation is needed. For the choice of the voltage/reactive power control \( u_Q \), we set \( u_Q = \pi_Q \) where \( \pi_Q \) is a constant vector enforcing the setpoint for the voltage dynamics. The role of this setpoint will be made clear in Subsection VI-B. Then, the main result of this section is as follows:

**Theorem 2.** Suppose that the vectors \( \theta^0 \in \mathbb{R}^n \) and \( V \in \mathbb{R}^n \) are such that Assumption 2 and condition (41), with \( \omega = \omega^* \), hold. Let \( u_P \) be given by (51), \( u_Q = \pi_Q \in \mathbb{R}^n \), and \( \pi_P \) the optimal input (48). Then, the following statements hold:

(i) The vector \( (D^T \theta, \omega, V, \xi) \) with \( (\theta, \omega, V, \xi) \) being a solution to (1), (51), with the conventional droop controller (14), quadratic droop controller (17), or reactive current controller (20), locally\(^5\) converges to the point

\(^5\)“locally” refers to the fact that the solutions are initialized in a suitable neighborhood of \( (\theta, \omega, V, \xi) \).
(\(D^T \theta^0, \overline{\varphi}, \overline{\psi}, \overline{\xi}\)).

(ii) The vector \((D^T \theta, \omega, V, \xi)\) with \((\theta, \omega, V, \xi)\) being a solution to (1), (51), with the E-ARP controller (23), locally converges to a point in the set

\[
\{(D^T \theta, \omega, V, \xi) \mid \omega = \omega^*, \xi = \pi_p, \quad P = \overline{\theta}, L_Q K_Q Q = K^{-1}_Q \pi_Q\}
\]

Moreover, for all \(t \geq 0\),

\[
1^T K^{-1}_Q \ln(V(t)) = 1^T K^{-1}_Q \ln(V(0)).
\]

**Proof:** First recall that \(\varphi = E^T \theta, \overline{\varphi} = E^T \overline{\theta}, \text{ and } D^T \overline{\varphi} = D^T \overline{\theta} = D^T \theta^0\) with \(E^T = [I_{n-1} - I_{n-1}]\) and noting that \(ED_1 = D\). By the compatibility property of the induced matrix norms, we have \(\|\varphi(0) - \overline{\varphi}\| \leq \|E^T \| \|\theta(0) - \overline{\theta}(0)\|\), thus showing that a choice of \(\theta(0)\) sufficiently close to \(\theta^0 = \overline{\theta}(0)\), returns an initial condition \(\varphi(0)\) sufficiently close to \(\overline{\varphi}\). We then consider a solution \((\theta(t), \omega(t), V(t), \xi(t))\) to the closed-loop system and express the solution in the new coordinates as \((\varphi(t), \omega(t), V(t), \xi(t))\).

Define the incremental storage function

\[
\mathcal{C}(\xi) = \frac{1}{2} \xi - \xi^T (\xi - \xi),
\]

with \(\xi = \pi_p\) given by (48). Then

\[
\frac{d}{dt} \mathcal{C} = -\xi^T (\xi - \xi)^T L_Q (\xi - \xi) - (\xi - \xi)^T (K^{-1}_P \omega - \pi_p)
\]

By (37), the time derivative of \(S(\theta(t), \omega(t), V(t))\) is equal to that of \(\mathcal{S}(\varphi(t), \omega(t), V(t))\), with \(\varphi(t)\) obtained from (34), namely (with (37) in mind)

\[
\frac{d}{dt} S(\theta(t), \omega(t), V(t)) = \frac{d}{dt} S(\varphi(t), \omega(t), V(t)).
\]

Hence, from the proof of Theorem 1 we infer that

\[
\frac{d}{dt} S(\varphi(t), \omega(t), V(t)) = -\xi^T (\xi - \xi)^T L_Q (\xi - \xi) - (\xi - \xi)^T (K^{-1}_P \omega - \pi_p)
\]

where \(X(V) = T_Q^{-1} K_Q [V], T_Q^{-1} \text{ or } [V] K_Q L_Q K_Q [V]\) and \(Y(V) = T_Q^{-1} T_Q^{-1} [V], [V]\) depending on the voltage controller adopted.

Observe that, by setting \(u_Q = \pi_Q\) and bearing in mind (54), the equalities (29), (31), (32), and (33) can be written in a unified manner as

\[
\frac{d}{dt} S(\varphi(t), \omega(t), V(t)) = -(\omega - \pi)^T K^{-1}_P (\omega - \pi) - \xi^T (\xi - \xi)^T L_Q (\xi - \xi)
\]

where \(X\) is a positive (semi)-define matrix defined above. Now taking \(S + \mathcal{C}\) as the Lyapunov function, we have

\[
\frac{d}{dt} S + \frac{d}{dt} \mathcal{C} = -(\omega - \pi)^T K^{-1}_P (\omega - \pi) - \xi^T (\xi - \xi)^T L_Q (\xi - \xi).
\]

By local strict convexity of \(S + \mathcal{C}\) (thanks to (41)), we can construct a forward invariant compact level set \(\Omega\) around \((\overline{\varphi}, \overline{\psi}, \overline{\xi})\) and apply LaSalle’s invariance principle. Notice in particular that on this forward invariant set we have \(V(t) \in R^n_+\) for all \(t \geq 0\). Solutions are then guaranteed to converge to the largest invariant set where

\[
\omega = \varpi, \quad 0 = L_P (\xi - \xi^*)
\]

The first equality yields \(\frac{dS}{d\omega} = 0\) on the invariant set \(\Omega\). Recall that \(\xi = \pi_p\). Hence, on the invariant set, \(L_P \xi = 0\) and thus \(\xi = \gamma I\) for some \(\gamma \in R\). Note that, by (51), \(\gamma\) has to be constant given the fact that \(\omega = \omega^*\) and \(L_P \xi = 0\). Also note that

\[
u_P = K_P (D^T \overline{\theta} V(\nu_P - \pi^*) = 0\)

on the invariant set. Multiplying both sides of the above equality by \(1^T K^{-1}_P\) yields \(\gamma 1^T K^{-1}_P \bar{\xi} = -1^T P^*\). Therefore, \(\xi = -\frac{1^T P^*}{1^T K^{-1}_P}\), and on the invariant set \(\Omega, \nu_P\) is equal to the optimal input \(\pi_P\) given by (48). This also means that \(\frac{dC}{d\xi} = 0\).

Notice that any solution \((\varphi(t), \omega(t), V(t), \xi(t))\) on the invariant set \(\Omega\) satisfies

\[
0 = -T^{-1}_P K_P E \frac{dS}{d\varphi} - T^{-2}_P K_P \frac{dS}{d\omega} + T^{-1}_P \frac{dC}{d\xi}.
\]

Hence, evaluating the dynamics above on the invariant set yields \(\frac{dS}{d\varphi} = 0\) noting that the matrix \(E\) has full column rank.

Furthermore, by (29), (31), and (32), the matrix \(X(V)\) is positive definite for the droop controller, quadratic droop controller, and the reactive current controller. Hence, the third equality in (56) yields \(\frac{dC}{d\xi} = 0\). Finally, by (57), the vector \(Q\) satisfies

\[
0 = T^{-1}_Q K_Q Q = T^{-1}_Q K_Q \pi_Q.
\]

By (27) and (57), the vector \(Q\) satisfies on the invariant set

\[
L_Q K_Q Q = K^{-1}_Q \pi_Q.
\]

Notice that, for the E-ARP controller, we have so far shown that the solutions \((\varphi(t), \omega(t), V(t), \xi(t))\) converge to the set

\[
Q := \{(\varphi(t), \omega(t), V(t), \xi(t)) \mid \omega = \omega^*, \xi = \pi_p, \quad P = \overline{\theta}, L_Q K_Q Q = K^{-1}_Q \pi_Q\}.
\]
Next, we establish the convergence of trajectories to a point in \( Q \). To this end, we take the forward invariant set \( \Omega \) sufficiently small such that
\[
\frac{\partial^2 (S + C)}{\partial (\phi, \omega, V, \xi)^2} > 0
\]
for every \((\phi, \omega, V, \xi) \in Q\). Note that this is always possible by (41) and continuity. Observe that any solution \((\phi, \omega, V, \xi)\) satisfies
\[
\begin{align*}
\dot{\phi} &= E^T K_P T^{-1} \frac{\partial S}{\partial \omega} \\
\dot{\omega} &= -T^{-1} K_P E \frac{\partial S}{\partial \phi} - T^{-2} K_P \frac{\partial S}{\partial \omega} + T^{-1} \frac{\partial C}{\partial \xi} \\
\dot{V} &= -T^{-1} X(V) \frac{\partial S}{\partial \phi} \\
\dot{\xi} &= -L_C \frac{\partial C}{\partial \xi} - K_P T^{-1} \frac{\partial S}{\partial \omega}.
\end{align*}
\]
It is easy to see that each point in \( Q \) is an equilibrium of the system above. Moreover, by (59), the incremental storage function \( S + C \) can be analogously defined with respect to any point in \( Q \) to establish Lyapunov stability by the inequality \( S + C \leq 0 \). Therefore, the positive limit set associated with any solution issuing from a point in \( Q \) contains a Lyapunov stable equilibrium. It then follows by [30, Proposition 4.7] that this positive limit set is a singleton which proves the convergence to a point in \( Q \). This proves the claim in the second statement of the theorem given the relationship between \( \phi \) and \( \omega \) variables exploited before.

Finally, by (27), the E-ARP controller can be written as
\[
\dot{V} = -[V] K_Q L_Q K_Q(Q - Q).
\]

Hence, we have
\[
\frac{d}{dt}(1^T K_Q^{-1} \ln V) = 1^T K_Q^{-1} [V]^{-1} [V] K_Q L_Q K_Q(Q - Q) = 0,
\]
as \( 1^T L_Q = 0 \), which proves that \( 1^T K_Q^{-1} \ln(V) \) is a conserved quantity.

Remark 8. (Stability under feedforward control) When the input \( u_P \) is set to the optimal feedforward input \( \pi_P \), rather than being generated by the feedback controller (51), the closed-loop system takes the form
\[
\begin{align*}
\dot{\phi} &= E^T K_P T^{-1} \frac{\partial S}{\partial \omega} \\
\dot{\omega} &= -T^{-1} K_P E \frac{\partial S}{\partial \phi} - T^{-2} K_P \frac{\partial S}{\partial \omega} \\
\dot{V} &= -T^{-1} X(V) \frac{\partial S}{\partial \phi}.
\end{align*}
\]
The same arguments as in the previous proof then show that solutions to this closed-loop system locally converge to the equilibrium point \((\phi, \omega, V, \xi)\). Hence, the stability of this equilibrium is an intrinsic property of the closed-loop system obtained setting \( u_P = \pi_P, u_Q = \pi_Q \). As mentioned before, the distributed integral controller (51) is adopted to overcome the lack of knowledge of \( \pi_P \), which depends on global parameters.

B. Power sharing

Theorem 2 portrays the asymptotic behavior of the microgrid models discussed in this paper, namely frequency regulation and voltage stability. In addition, optimal active power sharing for the coupled nonlinear microgrid model (1) is achieved if the droop coefficients \( K_P \) are suitably chosen. In fact, substituting (48) into (46) yields
\[
\mathcal{P} = P^* - K_P^{-1} \frac{1}{1^T K_P^{-1} 1},
\]
or, component-wise,
\[
\mathcal{P}_i = P_i^* - (k_P)_i^{-1} \frac{1}{1^T K_P^{-1} 1},
\]
where \( K_P = [k_P] \). In case droop coefficients are selected proportionally [46], [23], [2], [10], [52], i.e.
\[
(k_P)_i P_i^* = (k_P)_j P_j^*,
\]
for all \( i, j \), we conclude that
\[
(k_P)_i \mathcal{P}_i = (k_P)_j \mathcal{P}_j,
\]
which accounts for the desired proportional active power sharing based on the diagonal elements of \( K_P \) as expected.

Next, we take a closer look at other consequences and implications of Theorem 2 for different voltage dynamics.

1) Conventional droop controller: The vectors of voltages and reactive powers converge to \( \overline{V} \) and \( \overline{Q} \) satisfying
\[
K_Q \overline{Q} + \overline{V} = \pi_Q
\]
which yields
\[
\frac{(k_Q)_i \overline{Q}_i + \overline{V}_i}{(k_Q)_i \overline{Q}_j + \overline{V}_j} = \frac{(\pi_Q)_i}{(\pi_Q)_j}.
\]
This results in partial voltage regulation and reactive power sharing for the droop controlled inverters. In fact, for small values of \( k_Q \), the input \( \pi_Q \) regulates the voltages following (60). On the other hand, if the elements of \( k_Q \) are sufficiently large, reactive power is shared according to the elements of \( \pi_Q \) as given by (61). This tunable tradeoff between voltage regulation and reactive power sharing is consistent with the findings of [48].

2) Quadratic droop controller: The vector of voltages and reactive power converge to \( \overline{V} \) and \( \overline{Q} \) with
\[
K_Q [V]^{-2} + \overline{V} = \pi_Q.
\]
This implies that
\[
\frac{(k_Q)_i \overline{Q}_i + \overline{V}_i^2}{(k_Q)_j \overline{Q}_j + \overline{V}_j^2} = \frac{(\pi_Q)_i}{(\pi_Q)_j},
\]
which again results in a partial voltage regulation and reactive power sharing by an appropriate choice of \( k_Q \) and \( \pi_Q \). Moreover, in this case, the voltage variables at steady-state are explicitly given by
\[
\overline{V} = (I + K_Q A (\cos(D^T \theta_0))^{-1} \pi_Q).
\]
3) Reactive current controller: In this case, we have
\[ |\mathbf{V}|^{-1} \mathbf{Q} = \pi \mathbf{Q} \]
which results in
\[ \frac{\mathbf{Q}_i}{\mathbf{V}_i} = \left( \frac{\pi \mathbf{Q}_i}{\mathbf{Q}_i} \right)_j = \left( \frac{\mathbf{V}_j}{\mathbf{V}_i} \right) \left( \mathbf{Q}_i \right)_j. \]
The first equality provides the exact reactive current sharing, whereas the second equality can be interpreted as a mixed voltage and reactive power sharing condition. Moreover, the voltage variables at steady-state are given by
\[ \mathbf{V} = \mathbf{A}^{-1}(\cos(D^T \theta)) \pi \mathbf{Q}. \]

4) Exponentially-scaled averaging reactive power controller: In this case, the exact reactive power sharing can be achieved as evident from the second statement of Theorem 2, with \( \pi \mathbf{Q} = 0 \). In particular, by equality (58) with \( \pi \mathbf{Q} = 0 \) we obtain that
\[ \mathbf{Q} = \alpha K^{-1} \mathbf{1} \]
for some scalar \( \alpha \). Multiplying both sides of the above equality by \( \mathbf{1}^T \) yields
\[ \alpha = \frac{\mathbf{1}^T \mathbf{Q}}{\mathbf{1}^T K^{-1} \mathbf{1}}. \]
Clearly, \( \alpha > 0 \), by definition of \( \mathbf{Q} \) and as the matrix \( \mathbf{A} \) is positive definite. Therefore, as a consequence of Theorem 2, the vector of reactive power converges to a constant vector \( \mathbf{Q} \in \mathbb{R}^n_0 \) where
\[ (k_i) = (k_i) = (k_i) = (k_i), \]
which guarantees proportional reactive power sharing according to the elements of \( k_i \) as desired. Notice that the quantity \( \mathbf{1}^T K^{-1} \mathbf{1} \) is a conserved quantity in this case. Hence, the point of convergence for the voltage variables is primarily determined by the initialization \( \mathbf{V}(0) \).

VII. EXTENSIONS

In this section, we provide some extensions of the results established in the previous sections. In particular, we discuss microgrid stability and power sharing with lossy power lines, and with dynamic controllers \( \mathbf{u}_Q \).

A. Lossy lines

Under appropriate conditions, the stability of the system dynamics under the various controllers is preserved in the presence of lossy transmission lines that are homogeneous, namely whose impedances \( Z_{ij} \) equal \( |Z_{ij}|e^{\sqrt{-1}\phi} \), with \( \phi \in [0, \pi/2] \). Consistently, shunt components at the buses that are a series interconnection of a resistor and an inductor with impedance \( r_i + \sqrt{-1}x_i \) is considered. Assuming homogeneity of the shunt elements, i.e. \( r_i + \sqrt{-1}x_i = \sqrt{r_i^2 + x_i^2}e^{\sqrt{-1} \arctan \frac{x_i}{r_i}} = |Z_{ij}|e^{\sqrt{-1} \arctan \phi} \), where \( \phi = \arctan \frac{x_i}{r_i} \) for all \( i, \) routine derivations, see e.g. [61], [36], show that the total active and reactive power \( P_i, Q_i \) “exchanged” by the inverter \( i \) in the lossy network is equal to
\[ \begin{bmatrix} P_i \\ Q_i \end{bmatrix} = \Phi(\phi) \begin{bmatrix} P_i \\ Q_i \end{bmatrix}, \quad \Phi(\phi) = \begin{bmatrix} \sin \phi & \cos \phi \\ -\cos \phi & \sin \phi \end{bmatrix}, \]
where \( P_i, Q_i \), have the same expressions as in (2) and (3). Hence, the matrix \( \Phi(\phi) \) will modify the expressions of the active and reactive power exploited previously, and thus the frequency and voltage dynamics of the inverters will be changed accordingly, disrupting the convergence of the solutions. A natural way to counteract this modification is to exploit the inverse of \( \Phi(\phi) \) and use \( P^f \sin \phi - Q^f \cos \phi \) and \( P^f \cos \phi + Q^f \sin \phi \), with \( P^f = \text{col}(P_i^f) \) and \( Q^f = \text{col}(Q_i^f) \), in (1) instead of \( P \) and \( Q \), respectively. In this way, the lossless expressions of \( P_i, Q_i \) as in (2) and (3) will be recovered.

Notice that, the implementation of these controllers requires the knowledge of the parameter \( \phi \), which is assumed to be available. An interesting special case is obtained for \( \phi = 0 \), meaning that the network is purely resistive. In that case, in (1) \( P \) should be replaced by \( -Q \), and by \( P^f \), which is consistent with the use of droop controllers in resistive networks; see e.g. [7, Sec. II.A]).

As a result of the adaptation above, the same conclusions\(^4\) as in Theorem 2 hold for the lossy network with modified inverter dynamics. Notice, however, that the actual active power \( P^f \) will no longer be optimally shared in a lossy network with the conventional droop controller (14), quadratic droop controller (17), or the reactive current controller (20). Remarkably, in the case of the E-ARP controller, one can additionally prove that both active as well as reactive power sharing continues to hold. Because of its importance, this result is formalized below.

**Proposition 3.** For \( f(V, Q, u_Q) = -[V]K_QL_QK_QQ \), let Assumption 2 and condition (41) hold, with \( \Phi = \omega^* \) and \( \bar{B}_{ii}, B_{ij} \) replaced by \( |Z_{ii}|^{-1}, |Z_{ij}|^{-1} \), respectively. Then the vector \( (D^T \theta, \omega, V, \xi) \) a solution of
\[ \dot{\theta} = \omega \]
\[ T_P \dot{\omega} = -[(\omega - \omega^*) - K_P(P^f \sin \phi - Q^f \cos \phi - P^*) + u_P] \]
\[ V = -[V]K_QL_KQ(K^f \cos \phi + Q^f \sin \phi) \]
and \( u_P \) given by (51), locally converges to a point in the set
\[ \{(D^T \theta, \omega, V, \xi) \mid \omega = \omega^*, \xi = \pi_P, P = \bar{P}, L_QK_Q = 0 \}. \]

Moreover, \( \mathbf{1}^T K_Q^{-1} \ln(V(t)) = \mathbf{1}^T K_Q^{-1} \ln(V(0)) \), for all \( t \geq 0 \). Finally, \( P^f, Q^f \) converge to constant vectors \( \mathbf{T}_{P^f}, \mathbf{Q}_{Q^f} \) that satisfy
\[ (k_i) = (k_i) \]
\[ (k_j) = (k_j) \]
provided that
\[ \frac{(k_i)}{(k_i)} = \frac{(k_j)}{(k_j)}, \quad \forall i,j. \]

**Proof:** As remarked above, the convergence of the solutions is an immediate consequence of Theorem 2. Thus, we only focus on the power sharing property.

\(^4\)In these conditions, whenever relevant, the negative of the susceptances \( \bar{B}_{ii}, B_{ij} \) should be replaced by \( |Z_{ii}|^{-1}, |Z_{ij}|^{-1} \).
By condition (66) and relation (63) at steady state,
\[ P_i^f = (k_P) \frac{1}{j} P_j \sin \phi + (k_Q) \frac{1}{j} \bar{Q}_j \cos \phi = \frac{(k_P)}{(k_P)} P_i. \]

Similarly, for the reactive power
\[ Q_i^f = -P_i \cos \phi + \bar{Q}_i \sin \phi = -\frac{(k_P)}{(k_P)} Q_j \cos \phi + \frac{(k_Q)}{(k_Q)} Q_i \sin \phi = \frac{(k_Q)}{(k_Q)} Q_i. \]

### B. Dynamic extension

Another interesting feature is that thanks to the incremental passivity property the static controller \( u_Q = \bar{u}_Q \) can be extended to a dynamic controller. By Theorem 1 and keeping in mind Definition 1 together with (12) and (13), the incremental input-output pair of the voltage dynamics appears in the time derivative of the storage function \( S \) as
\[ (\frac{\partial S}{\partial V} - \frac{\partial S}{\partial V} \bigg|_{-})^T T_2^{-1} R_2 (u_Q - \bar{u}_Q), \]
where \( R_2 \) is the lower diagonal block of \( R \) in Theorem 1. Clearly this cross term vanishes by applying the feedforward input \( u_Q = \bar{u}_Q \). But an alternative way to compensate for this term is to introduce the dynamic controller
\[ T_Q \dot{\lambda} = -R_2 \frac{\partial S}{\partial V}, \]
\[ u_Q = K_\lambda \lambda, \]
for some positive definite matrix \( K_\lambda \). Notice that the controller above is decentralized for a diagonal matrix \( K_\lambda \), as the \( i \)th element of \( \frac{\partial S}{\partial V} \) is obtained from local variables \( V_i \) and \( Q_i \). Then, denoting the steady state value of \( \lambda \) by \( \bar{\lambda} \), the incremental storage function \( C_Q(\lambda) = \frac{1}{2} (\lambda - \bar{\lambda})^T K_\lambda (\lambda - \bar{\lambda}) \) satisfies
\[ \frac{d}{dt} C_Q = -(u_Q - \bar{u}_Q)^T T_2^{-1} R_2 \frac{\partial S}{\partial V}. \]
Recall that
\[ \frac{\partial S}{\partial V} = \frac{\partial S}{\partial V} - \frac{\partial S}{\partial V} \bigg|_{-}. \]
Therefore, (69) coincides with the negative of (67), and thus the same convergence analysis as before can be constructed based on the storage function \( S + C + C_Q \). Consequently, the result of Theorem 2 extends to the case of the dynamic voltage/reactive power controller (68). For illustration purposes, below we provide the exact expression of the controller above in case of the conventional droop controller:
\[ T_Q \dot{\lambda} = -[V]^{-1} K^{-1}_\lambda (K_Q(Q - \bar{Q}) + V - \bar{V}), \]
\[ u_Q = K_\lambda \lambda, \]
which by setting \( K_\lambda = K_Q \) reduces to
\[ T_Q u_Q = -[V]^{-1} (K_Q(Q - \bar{Q}) + V - \bar{V}). \]

Note that here the constant vectors \( \bar{V} \) and \( \bar{Q} \) are interpreted as the setpoints of the dynamic controller. It is easy to see that this controller rejects any unknown constant disturbance entering the voltage dynamics (14). Exploring other possible advantages of these dynamic controllers require further investigation, and is postponed to future research.
VIII. A NUMERICAL EXAMPLE

In this section, the performance of the previously discussed microgrid controllers is illustrated via a numerical example. We consider 4 inverters connected via lossless power lines. The topology of the grid is identified by the edge set \{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}. The susceptance of the lines are chosen as \( B_{12} = 4.17 \), \( B_{23} = 2.56 \), \( B_{24} = 1.67 \), and \( B_{34} = 6.25 \). The shunt susceptances are \( B_{11} = 9 \times 10^{-4} \), \( B_{22} = 7.5 \times 10^{-4} \), \( B_{33} = 6.38 \times 10^{-4} \), and \( B_{44} = 7.13 \times 10^{-4} \). The time constants \( T_P \) and \( T_Q \) are selected such that the response of different controllers have comparable settling time, and can be plotted conveniently below each other. The nominal frequency is equal to 50Hz, and the voltage and apparent power base values are selected as \( V_{base} = 20\text{KV} \) and \( S_{base} = 1\text{MVA} \), respectively. Active power setpoints are given by \( P^* = [2, -3.5, 1, 0.5]^T \text{(pu)} \). This indicates that the inverter at node 2 is operating in the charging mode and thus is treated as an active power load (see Remark 2). Reactive power setpoints are selected as \( Q^* = [0.48, 0.36, 0.24, 0.12]^T \text{(pu)} \). The droop coefficients for the active power are chosen as \( k_P = 0.75[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]^T \text{(pu)} \). Note that \((k_P)_{2}\) is chosen smaller than the other droop coefficients to penalize the generation at the second inverter such that it remains in the charging mode. For the droop and the quadratic droop controller, we set \( k_Q = 0.0208[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]^T \text{(pu)} \), and for the E-ARP controller droop coefficients are set as \( k_Q = 0.0038[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]^T \text{(pu)} \).

The system is initialized around the setpoints of the frequency, voltage magnitudes, active, and reactive power. At time \( t = 1\text{s} \), the active power setpoint of the load (node 2), is increased by 50 percent of its nominal value, resulting in voltage and frequency deviation from the nominal conditions. As can be seen from the top two plots in Figure 1, frequency is restored to 50Hz and the active power is shared proportionally according to the droop coefficients. Note that the frequency and active power response of the four voltage controllers does not differ much, and thus only the plot for the droop controller and active power response of the four voltage controllers does

Two major extensions can be envisioned. The first one is the investigation of similar techniques for network-preserved models of microgrids. Early results [17] show that this is feasible and will be further expanded in a follow-up publication. The second one is to use the obtained incremental passivity property to interconnect the microgrid with dynamic controllers, and obtain a better understanding of voltage control. Examples of these controllers are discussed in [48], but many others can be proposed and investigated. This, more generally, motivates the problem of identifying a class of voltage controllers which guarantees microgrid stability together with desired power sharing properties.

From a broader perspective, it is of interest to investigate how the set-up we have proposed can be extended to deal with other control problems that are formulated in the microgrid literature. Furthermore, the proposed controllers exchange information over a communication network, and it would be interesting to assess the impact of the communication layer on the results. In that regard, the use of Lyapunov functions is instrumental in advancing such research, since powerful Lyapunov-based techniques for the design of complex networked cyber-physical systems are already available; see e.g. [18]). Another intriguing question is to investigate possible relations between the role of Bregman distance in our stability analysis and convex optimization methods such as the mirror descent algorithm; see e.g. [62].

APPENDIX

Proof of Proposition 1. For the sake of notational simplicity, in this proof we omit the bar from all \( V, \varphi \). Clearly, the Hessian (38) is positive definite if and only if (44) holds. The latter is true if and only the matrix \( M \) below

\[
\begin{bmatrix}
\Gamma(V)[\cos(D_1^T \varphi)] & [\sin(D_1^T \varphi)]\Gamma(V)[D_1^T |V|^{-1}] \\
[V|^{-1}|D|\Gamma(V)[\sin(D_1^T \varphi)] & \mathcal{A}(\cos(D_1^T \varphi)) + \frac{\partial^2 H}{\partial V^2}
\end{bmatrix}
\]

is positive definite. In fact recall that the matrix in (44) can be written as the product

\[
\begin{bmatrix}
D_1 & 0 \\
0 & I
\end{bmatrix}
M
\begin{bmatrix}
D_1^T & 0 \\
0 & I
\end{bmatrix},
\]

and our claim descends from \( D_1D_1^T \) being nonsingular, the latter holding for \( D_1D_1^T \) is the principal submatrix of the Laplacian of a connected graph. Furthermore, note that by assumption \( \Gamma(V)[\cos(D_1^T \varphi)] \) is nonsingular. Then the Hessian is positive definite, or equivalently (44) holds, if and only if

\[
\Psi(D_1^T \varphi, V) := \mathcal{A}(\cos(D_1^T \varphi)) + [h(V)] - |V|^{-1}|D|\Gamma(V)[\sin(D_1^T \varphi)]^2[\cos(D_1^T \varphi)]^{-1}|D|^2|V|^{-1} > 0.
\]

Introduce the diagonal weight matrix, where \( \eta = D_1^T \varphi \),

\[
W(V, \eta) := \Gamma(V)[\sin(\eta)]^2[\cos(\eta)]^{-1}.
\]

For each \( k \sim \{i, j\} \in E \), its \( k \)th diagonal element is

\[
W_k(V_i, V_j, \eta_k) := B_{ij}V_iV_j\frac{\sin^2(\eta_k)}{\cos(\eta_k)}.
\]
Furthermore, it can be verified that

$$[D | \Gamma(V) [\sin(\eta)]^2 [\cos(\eta)]^{-1} | D | T]_{ij}$$

$$= \begin{cases} \sum_{k\sim\{i, j\} \in E} B_{id} V_i \frac{\sin^2(\eta_k)}{\cos(\eta_k)} & \text{if } i = j \\ B_{ij} V_i V_j \frac{\sin^2(\eta_k)}{\cos(\eta_k)} & \text{if } i \neq j, \end{cases}$$

from which

$$[\Gamma(V) [\sin(\eta)]^2 [\cos(\eta)]^{-1} | D | T [V]^{-1}]_{ij}$$

$$= \begin{cases} \sum_{k\sim\{i, j\} \in E} B_{id} V_i \frac{\sin^2(\eta_k)}{\cos(\eta_k)} & \text{if } i = j \\ \frac{\sin^2(\eta_k)}{\cos(\eta_k)} & \text{if } i \neq j. \end{cases}$$

On the other hand,

$$[A(\cos(\eta)) + [h(V)]]_{ij}$$

$$= \begin{cases} \dot{B}_{ii} + \sum_{k\sim\{i\} \in E} B_{ik} + h_i(V_i), & \text{if } i = j \\ -B_{ij} \cos(\eta_j) & \text{if } i \neq j \end{cases}$$

with $m \sim \{i, j\} \in E, i \neq j$. Suppose that each diagonal entry of matrix $\Psi(\eta, V)$ is positive, that is for each $i = 1, 2, \ldots, n$,

$$m_{ii} = \dot{B}_{ii} + \sum_{\ell=1, \ell \neq i}^n B_{i\ell} + h_i(V_i) - \sum_{k\sim\{i\} \in E} B_{id} \frac{V_i \sin^2(\eta_k)}{V_i \cos(\eta_k)}$$

$$= \dot{B}_{ii} + \sum_{k\sim\{i\} \in E} B_{id} \left(1 - \frac{V_i \sin^2(\eta_k)}{V_i \cos(\eta_k)} \right) + h_i(V_i) > 0.$$  

Notice that this holds true because of condition (42). Assume also that, for each $i = 1, 2, \ldots, n$,

$$m_{ii} > \sum_{k\sim\{i, \ell\} \in E} B_{id} \left| \cos(\eta_i) + \frac{\sin^2(\eta_k)}{\cos(\eta_k)} \right|$$

$$= \sum_{k\sim\{i, \ell\} \in E} B_{id} \sec(\eta_k),$$

which is condition (42). Then by Gershgorin theorem all the eigenvalues of the matrix $\Psi(\eta, V)$ have strictly positive real parts and the Hessian is positive definite.

**Proof of Proposition 2.** Under the given assumptions, by Lemma 2 there exists a vector $v = (v^{(1)}, v^{(2)}) \neq 0$ such that $v^T M v < 0$, where $v^{(1)}$ is the vector associated to the cut-set. Hence, it belongs to the cut-space, namely the column space of $D^T$ or equivalently to the one of $D^T$. As a result, bearing in mind (38), (44), the inequality $v^T M v < 0$ implies $w^T \frac{\partial^2 S}{\partial \gamma \partial \varphi} w < 0$ for some $w \neq 0$. In view of the symmetry of the Hessian, this in turn implies that the Hessian has a negative eigenvalue, thus proving the instability of $(\varphi, \varphi, \varphi)$ by Lemma 1.

**REFERENCES**


